Introduction

The Generalized Autoregressive Score (GAS) models, proposed by Creal et al. [1], are usually formulated as follows:

\[ y_t = \mu + \sqrt{\sigma_t} \varepsilon_t, \]
\[ f_t = \omega + ax_t + \text{esscher transform}, \]
where \( \varepsilon_t \) is the conditional innovation with zero mean and unit variance, \( f_t = \log p(y_t|y_{t-1}) \) is the conditional probability density function given parameter \( \theta \) and factor \( f_t \) is a scaling matrix. The idea behind the model is simple and intuitive: the latent variables evolve in the direction of maximum likelihood.

Our GAS-GH models (GAS with generalized hyperbolic innovations) incorporate conditional skewness and fat-tail, and generate realistic implied volatility smile. Based on the spirit of GJR-GARCH models, we propose Asymmetric GAS-GH models, which is:

\[ y_t = \mu + \sqrt{\sigma_t} \varepsilon_t, \]
\[ f_t = \omega + ax_t + \text{esscher transform}, \]
\[ f_t = \omega + ax_t + \text{esscher transform}, \]
where \( \varepsilon_t \) is the conditional innovation with zero mean and unit variance, \( S_t \) is a scaling matrix and \( f_t \) is the asset return.

Methodology

The GAS model can be estimated by Maximum Likelihood Estimation (MLE), and for the change of measure, we assume a class of probability transformations is specified as follows:

\[ Q^\theta(dx) = \frac{c^{\theta}(\mu, \sigma, \alpha, \beta, \mu, x_0)}{c^{\theta}(\mu, \sigma, \alpha, \beta, \mu, x_0)} dx. \]

Under risk neutral measure, the conditional expected return of a risky asset should be equal to the risk free rate, which means:

\[ E^Q[y_t|F_t] = r, \]
where \( r \) is the risk-free rate. Thus, the classical Escher Transform parameter can be solved by the equation above.

The detailed option pricing procedure is as follows:

1. Estimate the model with the historical returns from time \( t-N \) to \( t \).
2. Filter out the volatility, \( \sqrt{\sigma_t} \) according to the estimated parameters from the last step.
3. Obtain the asset return process under risk neutral measure and then simulate \( M \) asset return paths from time \( t \) to time \( t+1 \), obtain \( M \) price paths \( [P_t]_{t=1,2,\ldots,M} \).
4. Compute the option price as follows:

\[ C_t = \frac{1}{M} \sum_{i=1}^{M} c_i \left( -\frac{\Delta T}{V} \right), \]

where \( V \) is the pay-off function which may be path dependent, and \( C_t \) is the price of the option.

Option Pricing Models

For comparison, we use GARCH class models and GAS-GH class models, and derive their risk-neutral parameters with Escher transform.

First, standard GARCH models are formulated as follows:

\[ y_t = \mu + \sqrt{\sigma_t} \varepsilon_t, \]
\[ h_{t+1} = \omega + a(S_t + \text{esscher transform}), \]
where \( h_t \) is the conditional variance of the log return.

After implementing Escher transform, we obtain the distribution of asset return under risk-neutral measure. Under risk-neutral measure, the model is:

\[ R_t = r - \frac{1}{2} \sqrt{\sigma_t} \varepsilon_t, \]

\[ \text{Similarly, we could also derive the Heston and Nandi (2000) GARCH model.} \]

For the class of generalized hyperbolic (GH) distribution models, we have:

\[ y_t = \mu + \sqrt{\sigma_t} \varepsilon_t, \]
\[ f_t = \omega + ax_t + \text{esscher transform}, \]
where \( \varepsilon_t \) is a GH random variable with zero mean and unit variance, \( S_t \) is a scaling matrix and \( f_t \) is the asset return.

The GH distributed random variable has the following representation form:

\[ \varepsilon_t = \mu + \gamma + \sqrt{\chi} X, \]
where \( X \) is distributed with pdf:

\[ p(\chi,\gamma,\varepsilon_t) = C^{-1} \exp \left( \frac{1}{2} \chi^{-1} + \psi \right). \]

Estimation and Simulation

We first estimate each model by MLE, then price the options by simulation. Based on the estimates, we implement Escher transform to convert the physical measure to risk neutral measure. Then we obtain the option price by simulation. The initial volatility input is the sample annualized volatility from January 1962 to August 2014, which is 19.41%. The spot price and risk-free rate are assumed to be 100 and 0 respectively. The number of simulation trials is 200000. The figures below shows the Implied Volatility Surface (IVS) generated by each model.

Figure 1: Simulated Implied Volatility Surface

Empirical Analysis

The S&P 500 option price data are from OptionMetrics database. We eliminate options with zero trading volume and only keep the out-of-the-money options and options with \( F/K > 1.2 \). The testing period is from January 2014 to August 2014. The testing procedure is as follows:

1. Use the S&P 500 daily returns from \( t_0 \) to \( t \) to estimate the model parameters, and then filter out the volatility \( \sigma_t \) of the model.
2. Given the underlying price \( S_t \) and initial volatility input \( \sigma_{t_0} \), obtain the option price generated by the model through simulation and then compute the implied volatility of each option.

After computing the implied volatility generated by the model of all the options in the sample period, we can obtain the square mean error:

\[ \text{IVMSE} = \sqrt{\frac{1}{N} \sum (\text{ImpliedVolatility} - \text{OptionVolatility})^2}, \]

where \( N \) is the total number of option contracts in the sample period.

Table 1: Option Pricing Results

We find that the asymmetric GAS variance gamma model performs well when pricing OTM put options. Overall speaking, it also achieves the best performance of all the option pricing models.

Conclusion

Following Zhu and Ling [2], we develop a model based option pricing procedure for GAS models and investigate the option pricing performance of GAS models with generalized hyperbolic innovations. Besides, we propose a new GAS-GH model where the dynamics of volatility depends on the sign of spot returns. Simulation studies show that GAS-GH models can produce a realistic implied volatility smile. After fitting each model with data, we found that asymmetric GAS-variance gamma model performs best within our sample period. In the paper, we also analysis the option pricing performance of GAS-Jump models and now are in the progress of Semi-parametric GAS models.

Main References
